

# Degenerating abelian varieties via log abelian varieties

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**ABSTRACT.** For any split totally degenerate abelian variety over a complete discrete valuation field, we construct a log abelian variety, in the sense of [KKN08a], over the discrete valuation ring extending the given abelian variety. This generalises Kato's Tate curve.

## Introduction

Degeneration appears naturally in compactifications of moduli spaces. Usually we prefer compactifications coming from moduli problems, in other words we prefer to use canonical (in suitable sense) degenerate objects to make compactifications.

In the theory of classical toroidal compactifications of the moduli spaces of abelian varieties, there is no canonical choice of toroidal degenerations of abelian varieties. In late 80's, Kato [Kat89, Sec. 2.2] formulated a construction of log Tate curve, and conjectured the existence of a general theory of log abelian varieties. Later Kato and his co-authors realised the theory of log abelian varieties in [KKN08b, KKN08a]. Note as indicated in [KKN08a], there are other constructions of log abelian varieties in [Pah05, Ols03]. However in this paper, we stick to the one defined in [KKN08a]. In some sense, a log abelian variety as a degeneration of a given abelian variety is to treat all possible toroidal degenerations of that abelian variety as “one object”, hence it becomes canonical. This “one object” is proper, smooth, and even has a group structure on itself in the world of log geometry. These aspects make log abelian variety a perfect degeneration of abelian variety. For application of log abelian varieties, the short exact sequence in [KKN08a, 4.1.2] is the upshot.

Let  $R$  be a complete discrete valuation ring with fraction field  $K$ , let  $A_K$  be an abelian variety over  $K$ . Since log abelian varieties are supposed to be canonical degenerations of abelian varieties, there should be a canonical (unique) log abelian variety  $A$  over  $R$  extending  $A_K$ . As an example of log abelian variety, the authors of [KKN08a] constructed such a log abelian variety  $\mathcal{E}_q$  over  $R$  (or  $O_K$  as in their notation) for the Tate curve  $E_q$  over  $K$  with “ $q$ -invariant”  $q$ , see [KKN08a, 1.6, 1.7, 4.7]. In this paper we generalise their log Tate curve to higher dimension

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case for split totally degenerate abelian varieties over complete discrete valuation fields. The difficulty of the generalisation lies in two aspects. Firstly, in the curve case, the formal toroidal models can always be algebraized to schemes, whilst the higher dimension case we have to turn to algebraic spaces which are more technical. Secondly, as in most cases in mathematics, hard combinatorics shows up in higher dimension.

In the first section, we give the setting-up. In section 2, the main result is 2.2, which says that the formal toroidal model  $\mathcal{A}_\Sigma$  associated to any  $Y$ -admissible polytope decomposition  $\Sigma$  algebraizes to an algebraic space  $A_\Sigma$ . Artin's theorems on "existence of contractions and dilatations" [Art70] are crucial for the proof. In section 3, we investigate the algebraic space  $A_\Sigma$  in some details, and put a canonical log structure on it. The key point of this section is Corollary 3.2. In section 4, we give the construction (4.3) of  $A$ , and show that  $A$  is the log abelian variety extending the given abelian variety  $A_K$  over  $K$  in theorem 4.2. And the association of log abelian variety  $A$  to  $A_K$  is actually a functor, see theorem 4.3.

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### 1. Setting-up

Let's work with  $S = \operatorname{Spec} R$ , where  $R$  is a complete DVR with fraction field  $K$ , uniformiser  $\pi$  and residue field  $k$ , let  $S_n = \operatorname{Spec} R/(\pi)^{n+1}$  for  $n \in \mathbb{N}$ , we also use the notation  $s$  for  $S_0$ . We regard  $S$  and  $S_n$  as log schemes with respect to the canonical log structures, and let  $i_n : S_n \rightarrow S$  be the inclusion. Let  $j$  be the open immersion  $\operatorname{Spec} K \rightarrow S$ , and we also write  $i_0$  as  $i$ .

Let  $(\operatorname{fs}/S)$  be the category of fs log algebraic spaces over  $S$ , and we regard it as a site endowed with the classical étale topology. Let  $(\operatorname{fs}/S)'$  be the full subcategory of  $(\operatorname{fs}/S)$  consisting of objects on which  $\pi$  is locally nilpotent. We also endow  $(\operatorname{fs}/S)'$  with the classical étale topology. For any fs log algebraic space  $X$  over  $S$ , we don't distinguish the log algebraic space  $X$  from the sheaf on  $(\operatorname{fs}/S)$  represented by  $X$ .

Let  $A_K$  over  $K$  be a semi-stable abelian variety of dimension  $d$  and let  $A_K^*$  be the dual abelian variety of  $A_K$ , then we have the following two diagrams

$$(1.1) \quad \begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & Y & & & & \\ & & \vdots & \searrow v & & & \\ & & \downarrow u_K & & & & \\ 0 & \longrightarrow & T & \longrightarrow & \tilde{G} & \longrightarrow & B \longrightarrow 0 \\ & & & & \vdots & & \\ & & & & A_K & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

and

$$(1.2) \quad \begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & X & & & & \\ & & \vdots & \searrow v^* & & & \\ & & \downarrow u_K^* & & & & \\ 0 & \longrightarrow & T^* & \longrightarrow & \tilde{G}^* & \longrightarrow & B^* \longrightarrow 0 \\ & & & & \vdots & & \\ & & & & A_K^* & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

The diagrams (1.1) and (1.2) are explained as follows:

- (a) the rows in (1.1) and (1.2) are exact sequences of group schemes over  $S$ , which are the Raynaud extensions associated to  $A_K$  and  $A_K^*$  respectively. In particular,  $T$  and  $T^*$  are tori over  $S$ , and  $B$  and  $B^*$  are abelian schemes over  $S$ ;
- (b) the morphisms (labeled as dashed arrows) in the columns in (1.1) and (1.2) are defined rigid-analytically over  $K$ , but the morphisms  $u_K$  and  $u_K^*$  are also algebraic;  $Y$  (resp.  $X$ ) is the character group of  $T^*$  (resp.  $T$ ) which is a locally constant sheaf over  $S$  represented by a finite rank free  $\mathbb{Z}$ -module étale locally, and  $\tilde{G}_K = \tilde{G} \times_S K$  (resp.  $\tilde{G}_K^* = \tilde{G}^* \times_S K$ ) is the rigid analytic uniformization of  $A_K$  (resp.  $A_K^*$ );

- (c)  $v$  (resp.  $v^*$ ) is a morphism of group schemes over  $S$  given by the 1-motive dual of the Raynaud extension associated to  $A_K^*$  (resp.  $A_K$ ), and  $u_K^*$  (resp.  $u_K'$ ) lifts  $v$  (resp.  $v^*$ ) over  $K$ .

Via the duality theory of 1-motives, the diagram (1.1) (or equivalently (1.2)) is equivalent to another commutative diagram

$$(1.3) \quad \begin{array}{ccccccc} & & & & Y \times X & & \\ & & & \swarrow s_K & \downarrow v \times v^* & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathcal{P}_B & \longrightarrow & B \times B^* \longrightarrow 0 \end{array}$$

where the row in the diagram is the Poincaré biextension of  $(B, B^*)$  by  $\mathbb{G}_m$  and  $s_K$  is a bilinear section over  $K$  along  $v \times v^*$ .

From now on, we assume that  $T$  is a split torus, and  $A_K$  is totally degenerate, i.e.  $B$  is zero (in future we will deal with the general case). And in this case, we say  $A_K$  is spit totally degenerate. Then the bilinear section  $s_K$  is just a bilinear pairing

$$(1.4) \quad \langle, \rangle: X \times Y \rightarrow K^\times$$

here we switch the positions of  $Y$  and  $X$  for coincidence with [KKN08a]).

## 2. Constructing proper model $A_\Gamma$ of $A_K$ associated to a polytope decomposition $\Gamma$

As in [Mum72, Section 6], we study the (convex) polytope decompositions of the affine space  $E := \text{Hom}(X, \mathbb{Q})$ . For the notions concerning polytopes, we refer to [Oda88, Appendix]

**DEFINITION 2.1.** A **polytope decomposition**  $\Sigma$  of  $E$  is a set of polytopes  $\sigma \subset E$  such that

- (1)  $\cup_{\sigma \in \Sigma} \sigma = E$ ;
- (2) if  $\tau \leq \sigma$  and  $\sigma \in \Sigma$ , then  $\tau \in \Sigma$ ;
- (3) if  $\sigma, \tau \in \Sigma$  with  $\sigma \cap \tau \neq \emptyset$ , then  $\sigma \cap \tau$  is a common face of  $\sigma$  and  $\tau$ .

Given another polytope decomposition  $\Sigma'$  of  $E$ , there exists a map from  $\Sigma'$  to  $\Sigma$  if for any  $\sigma' \in \Sigma'$  there is a  $\sigma \in \Sigma$  such that  $\sigma' \subseteq \sigma$ . It is easy to see if such a map exists, it is unique and realises  $\Sigma'$  as a subdivision of  $\Sigma$ .

Let  $H$  be a group acting on  $E$ , a decomposition  $\Sigma$  is called  **$H$ -stable**, if  $h \cdot \sigma \in \Sigma$  for any  $\sigma \in \Sigma$  and  $h \in H$ . If moreover  $\Sigma$  has only finitely many orbits, then  $\Sigma$  is called  **$H$ -admissible** (or simply **admissible** if the underlying group is clear in context).

If  $\Sigma'$  is  $H'$ -stable for another group  $H'$  which acts on  $E$  too, and we are given a homomorphism from  $H'$  to  $H$ , then a map from  $\Sigma'$  to  $\Sigma$  is **equivariant**, if it is compatible with the group actions (we will be particularly interested in the case that  $H'$  is a subgroup of  $H$ ).

Via the bilinear form  $\langle, \rangle: X \times Y \rightarrow K^\times$ , see (1.4), we get a natural action of  $Y$  on  $T_K = \text{Hom}(X, \mathbb{G}_{m,K})$ , hence on  $E = \text{Hom}(X, \mathbb{Q})$ . We will mostly work with

$Y$ -action on  $E$ , hence being admissible will always mean being  $Y$ -admissible if the acting group is not specified.

**FACT 2.1.** *Given any admissible polytope decomposition  $\Sigma$  of  $E$ , by [KKMSD73, Chap. IV, sect. 3] and [Mum72, Cor. 6.6] we get a normal scheme  $P_\Sigma$  locally of finite type over  $S$  such that:*

- (a)  $P_{\Sigma,K} = T_K$ .
- (b) *The translation action of  $T$  extends to  $P_\Sigma$  and  $P_\Sigma$  can be covered by some  $T$ -invariant affine open sets  $P_\sigma$ , which are in one to one correspondence with  $\sigma \in \Sigma$ . Here  $P_\sigma = \text{Spec} A_\sigma$ , where  $A_\sigma = R[C(\sigma)^\vee \cap \mathbb{X}']$ ,  $\mathbb{X}' = \pi^\mathbb{Z} \oplus X$  and  $C(\sigma)$  is the cone in  $\mathbb{E} = \text{Hom}(\mathbb{X}', \mathbb{Q}) = \mathbb{Q} \oplus E$  above  $\sigma \subseteq E$  with  $E$  identified with the hyperplane  $(1, E)$  in  $\mathbb{E}$ .*
- (c)  $P_\sigma \cap P_\tau = P_{\sigma \cap \tau}$  (resp.  $P_\sigma \cap P_\tau = T_K$ ) for any  $\sigma, \tau \in \Sigma$  with  $\sigma \cap \tau \neq \emptyset$  (resp.  $\sigma \cap \tau = \emptyset$ ).
- (d) *The torus  $T$  naturally embeds into  $P_\Sigma$  if and only if  $\{0\} \in \Sigma$ .*
- (e) *For all valuations  $v$  on  $\text{Frac}(T)$  if  $v \geq 0$  on  $R$  and if [for any  $\alpha \in X, \exists n \in \mathbb{Z}$  such that  $n \cdot v(\pi) \geq v(\mathcal{X}^\alpha) \geq -nv(\pi)$ ] hold, then  $v$  has a centre on  $P_\Sigma$ .*
- (f) *The action of  $Y$  on  $\Sigma$  gives rise to an action of  $Y$  on  $P_\Sigma$ , via*

$$S_y : P_\sigma \rightarrow P_{y+\sigma}$$

$$S_y^* : C(y + \sigma)^\vee \cap \mathbb{X}' \rightarrow C(\sigma)^\vee \cap \mathbb{X}', \pi^n x \mapsto \langle x, y \rangle \pi^n x$$

for  $y \in Y$  and  $\pi^n x \in C(y + \sigma)^\vee \cap \mathbb{X}'$  with  $x$  the  $X$ -part. And the action induces an action on  $P_{\Sigma,n} := P_\Sigma \times_S S_n$  for each positive integer  $n$ .

If  $\tilde{\Sigma}$  is a  $H$ -admissible subdivision of  $\Sigma$  for  $H$  a subgroup of  $Y$ , then we have a natural morphism of  $S$ -schemes  $P_{\tilde{\Sigma}} \rightarrow P_\Sigma$  which is compatible with the group actions.

**PROPOSITION 2.1.** *The quotient of the  $Y$ -action on  $P_{\Sigma,n}$  exists in the category of schemes over  $S_n$ .*

**PROOF.** This is obvious, since  $P_{\sigma,n} \cap P_{\tau,n} = P_{\sigma \cap \tau, n}$  (resp.  $P_{\sigma,n} \cap P_{\tau,n} = \emptyset$ ) for  $\sigma, \tau \in \Sigma$  with  $\sigma \cap \tau \neq \emptyset$  (resp.  $\sigma \cap \tau = \emptyset$ ).  $\square$

Now we can formulate the quotient scheme of  $P_{\Sigma,n} = P_\Sigma \times_S S_n$  by the action of  $Y$ , denoted by  $A_{\Sigma,n}$ . Taking the colimit of  $A_{\Sigma,n}$ , we get a formal scheme  $\mathcal{A}_{Y,\Sigma}$  over  $\mathcal{S} = \text{Spf} R$ . We use the simple notation  $\mathcal{A}_\Sigma$  instead of  $\mathcal{A}_{Y,\Sigma}$  if no other group action is involved.

Taking another  $H$ -admissible polytope subdecomposition  $\tilde{\Sigma}$  of  $\Sigma$  into account, we get a morphism  $\mathcal{A}_{H,\tilde{\Sigma}} \rightarrow \mathcal{A}_{H,\Sigma}$  of  $\mathcal{S}$ -formal schemes. If  $H = Y$ , we simply write  $\mathcal{A}_{\tilde{\Sigma}} \rightarrow \mathcal{A}_\Sigma$ .

It follows that we get a functor

$$(2.1) \quad \mathcal{F} : \{(H, \Gamma)\} \rightarrow \{\mathcal{S}\text{-formal schemes}\}$$

from the category of pairs  $(H, \Gamma)$ , where  $H$  is a subgroup of  $Y$  and  $\Gamma$  is a  $H$ -admissible polytope decomposition of  $E$ , to the category of formal schemes over  $\mathcal{S}$ .

This functor restricts to the category of  $Y$ -admissible polytope decompositions of  $E$ .

**THEOREM 2.1** (Mumford[Mum72], Alexeev and Nakamura [AN99], Alexeev[Ale02]). *There exists a  $Y$ -admissible polytope decomposition  $\Xi$  such that the formal scheme  $\mathcal{A}_\Xi$  admits an ample line bundle, hence it is algebraisable, i.e. it comes from the formal completion of a unique algebraic scheme  $A_\Xi$  over  $S$  along its special fibre.*

*Moreover,  $A_\Xi$  is a stable semiabelic scheme under the semiabelian scheme  $G$  over  $S$ , where  $G$  comes from the semistable reduction theorem. And we can choose  $\Xi$  such that  $\{0\} \in \Xi$ , hence  $A_\Xi$  contains  $G$  as an open subscheme canonically.*

From now on, we fix such a  $Y$ -admissible polytope decomposition  $\Xi$ . The main result in this section is the following theorem.

**THEOREM 2.2** (Algebraisation of formal models). *Let  $\Gamma$  be a  $Y$ -admissible polytope decomposition of  $E$ , the formal scheme  $\mathcal{A}_\Gamma$  over  $S$  comes from the formal completion of a unique algebraic space  $A_\Gamma$  over  $S$  along its special fibre.*

*Moreover,  $A_\Gamma$  is a proper model (in the category of  $S$ -algebraic spaces) of  $A_K$ , i.e. the structure morphism of  $A_\Gamma$  over  $S$  is proper and the generic fibre  $(A_\Gamma)_K = A_\Gamma \times_S K$  coincides with  $A_K$ .*

**COROLLARY 2.1.** *Let  $\text{PolDecom}_Y$  be the category of pairs  $(H, \Gamma)$  as in (2.1) with the extra condition that  $H$  is of finite index in  $Y$ . Then we have a functor*

$$(2.2) \quad M : \text{PolDecom}_Y \rightarrow \{\text{proper algebraic } S\text{-spaces}\}$$

*from  $\text{PolDecom}_Y$  to the category of proper algebraic spaces over  $S$ , such that the functor  $\mathcal{F}$  in (2.1) restricting to  $\text{PolDecom}_Y$  factors through  $M$ .*

**PROOF.** This follows from theorem 2.2. □

We will denote the algebraic  $S$ -space  $M((H, \Gamma))$  as  $A_{H, \Gamma}$ , and if  $H = Y$  we simply use the notation  $A_\Gamma$ .

The main ingredient of the proof of theorem 2.2 is Artin's "existence of contractions" and "existence of dilatations" theorems, see [Art70, 3.1, 3.2]. We start with the following proposition.

**PROPOSITION 2.2.** *Given a map  $\iota : \Sigma \rightarrow \Gamma$  between two  $Y$ -admissible polytope decompositions of  $E$  (note  $\iota$  has to be a  $Y$ -admissible subdivision), the morphism*

$$\mathcal{F}(\iota) : \mathcal{A}_\Sigma \rightarrow \mathcal{A}_\Gamma$$

*of  $S$ -formal schemes is a formal modification in the sense of [Art70, 1.7].*

**PROOF.** We need to show that  $\mathcal{F}(\iota)$  is proper and verifies the three conditions in [Art70, Definition (1.7)].

It's enough to show  $\mathcal{F}(\iota)_0 : P_{\Sigma,0}/Y \rightarrow P_{\Gamma,0}/Y$  is proper, and we use the valuative criterion. Given any commutative diagram

$$(2.3) \quad \begin{array}{ccc} \eta & \xrightarrow{\alpha} & P_{\Sigma,0}/Y \\ j \downarrow & & \downarrow \mathcal{F}(\iota)_0 \\ V & \xrightarrow{\beta} & P_{\Gamma,0}/Y \end{array}$$

with  $V$  the spectrum of a discrete valuation ring and  $\eta$  the open point of  $V$ . Then there exists a  $\bar{\sigma} \in \Sigma/Y$  (resp.  $\bar{\gamma} \in \Gamma/Y$ ) such that  $\alpha(\eta)$  (resp.  $\beta(\eta)$ ) lies in the corresponding  $T_0$ -orbit in  $P_{\Sigma,0}/Y$  (resp.  $P_{\Gamma,0}/Y$ ). Then we have that

$$\overline{O_{\bar{\sigma}}} = \bigcup_{\{\bar{\sigma}_i \in \Sigma/Y \mid \bar{\sigma} \subset \bar{\sigma}_i\}} O_{\bar{\sigma}_i}, \quad \overline{O_{\bar{\gamma}}} = \bigcup_{\{\bar{\gamma}_j \in \Gamma/Y \mid \bar{\gamma} \subset \bar{\gamma}_j\}} O_{\bar{\gamma}_j}$$

and the morphism  $\beta$  factors through  $\overline{\beta(\eta)}$ . Choose suitable liftings  $\sigma_i$ 's (resp.  $\gamma_j$ 's) of  $\bar{\sigma}_i$ 's (resp.  $\bar{\gamma}_j$ 's) such that all  $\sigma_i$ 's (resp.  $\gamma_j$ 's) contain the lifting  $\sigma$  (resp.  $\gamma$ ) and  $\cup_i \sigma_i \subset \cup_j \gamma_j \subset E$ . Then the diagram (2.3) lifts to a commutative diagram

$$(2.4) \quad \begin{array}{ccc} \eta & \xrightarrow{\tilde{\alpha}} & P_{\Sigma,0} \\ j \downarrow & \nearrow \tilde{\delta} & \downarrow \\ V & \xrightarrow{\tilde{\beta}} & P_{\Gamma,0} \end{array}$$

in which the morphism  $P_{\Sigma,0} \rightarrow P_{\Gamma,0}$  is proper, hence  $\tilde{\alpha}$  factors through some  $\tilde{\delta}$ . And  $\tilde{\delta}$  factors through  $\bigcup_{\{\sigma_i\} \subset \Sigma} O_{\sigma_i}$ , so gives rise to a morphism  $\delta : V \rightarrow P_{\Sigma,0}/Y$ . It's easy to see that  $\alpha$  factor through  $\delta$ . On the other hand if there exists another morphism  $\delta'$  such that  $\alpha = \delta' \circ j$ , then we can lift  $\delta'$  to a morphism  $\tilde{\delta}' : V \rightarrow P_{\Sigma,0}$  such that  $\tilde{\alpha} = \tilde{\delta}' \circ j$ . The properness of  $P_{\Sigma,0} \rightarrow P_{\Gamma,0}$  implies  $\tilde{\delta}' = \tilde{\delta}$ , hence  $\delta = \delta'$ . Then the properness of  $\mathcal{F}(\iota)_0$  follows.

We have a morphism  $P_{\Sigma} \rightarrow P_{\Gamma}$  of  $\mathcal{S}$ -schemes induced by  $\iota$ . Let  $\mathcal{P}_{\Sigma} = \varinjlim_n P_{\Sigma,n}$  and  $\mathcal{P}_{\Gamma} = \varinjlim_n P_{\Gamma,n}$ , then we have a morphism  $\mathcal{P}_{\Sigma} \rightarrow \mathcal{P}_{\Gamma}$  of  $\mathcal{S}$ -formal schemes, which we still denote by  $\iota$ . By [Art70, Corollary (1.15)],  $\iota$  is the formal modification induced by  $P_{\Sigma} \rightarrow P_{\Gamma}$ . We have the following Cartesian diagram

$$\begin{array}{ccc} \mathcal{P}_{\Sigma} & \longrightarrow & \mathcal{A}_{\Sigma} \\ \iota \downarrow & & \downarrow \mathcal{F}(\iota) \\ \mathcal{P}_{\Gamma} & \longrightarrow & \mathcal{A}_{\Gamma} \end{array}$$

with the rows étale coverings. Since the notion of formal modification is local on the base for the (formal) étale topology (see [Art70, sixth line of the proof of Proposition (1.13)]),  $\mathcal{F}(\iota)$  is a formal modification.  $\square$

**PROOF OF THEOREM 2.2:** Let  $\Xi \sqcap \Gamma$  be, similar as in [KKN08b, 5.2.15], the set of polytopes of the form  $\xi \cap \gamma$  for  $\xi \in \Xi, \gamma \in \Gamma$ . Then we have the following

diagram

$$(2.5) \quad \begin{array}{ccc} & \Xi \sqcap \Gamma & \\ \swarrow & & \searrow \\ \Xi & & \Gamma \end{array}$$

in the category of  $Y$ -admissible polytope decompositions, hence the following diagram

$$(2.6) \quad \begin{array}{ccc} & \mathcal{A}_{\Xi \sqcap \Gamma} & \\ \swarrow & & \searrow \\ \mathcal{A}_{\Xi} & & \mathcal{A}_{\Gamma} \end{array}$$

in the category of  $\mathcal{S}$ -formal schemes with the arrows being formal modifications. We know that  $\mathcal{A}_{\Xi}$  algebraizes to an  $S$ -scheme  $A_{\Xi}$ . By [Art70, 3.2] and [Art70, 3.1], we get the following diagram

$$(2.7) \quad \begin{array}{ccc} & A_{\Xi \sqcap \Gamma} & \\ \swarrow & & \searrow \\ A_{\Xi} & & A_{\Gamma} \end{array}$$

of algebraic  $S$ -spaces, where  $A_{\Xi \sqcap \Gamma}$  and  $A_{\Gamma}$  are the algebraizations of  $\mathcal{A}_{\Xi \sqcap \Gamma}$  and  $\mathcal{A}_{\Gamma}$  respectively, and the two arrows are the corresponding modifications associated to the formal modifications in diagram (2.6). The properness of  $A_{\Gamma}$  over  $S$  follows from the properness of  $A_{\Xi}$  and the properness of modifications. Since  $(A_{\Xi})_K = A_{\Xi} \times_S K$  is isomorphic to  $A_K$ , so is  $(A_{\Gamma})_K$ .  $\square$

Given an admissible polytope decomposition  $\Gamma$ , and a subset  $\bar{\Gamma}_0 \subset \bar{\Gamma}$  which is stable under taking faces (i.e.  $\bar{\gamma} \in \bar{\Gamma}$  and  $\bar{\tau} \leq \bar{\gamma}$  imply that  $\bar{\tau} \in \bar{\Gamma}_0$ ), we can define an open algebraic subspace  $A_{\Gamma, \bar{\Gamma}_0}$  of  $A_{\Gamma}$  as follows. Note giving  $\bar{\Gamma}_0$  is the same as giving a subset  $\Gamma_0 \subset \Gamma$  which is  $Y$ -stable and stable under taking faces. Then we have a closed subset

$$\tilde{B} = \bigcup_{\gamma \in \Gamma \setminus \Gamma_0} O_{\gamma} \subset P_{\Gamma}$$

and make it into a reduced closed subscheme of  $P_{\Gamma}$ . The closed formal subscheme  $\varinjlim_n (\tilde{B} \times_S S_n)/Y$  of  $\varinjlim_n P_{\Gamma, n}/Y$  gives rise to a closed algebraic subspace  $B$  of  $A_{\Gamma}$ . Taking the complement of  $B$ , we get the open algebraic subspace  $A_{\Gamma, \bar{\Gamma}_0}$ . In the case that  $\bar{\Gamma}_0 = \{\bar{\tau} \mid \tau \leq \gamma\}$  for some  $\gamma \in \Gamma$ , we also use the notation  $A_{\Gamma, \bar{\gamma}}$  for  $A_{\Gamma, \bar{\Gamma}_0}$ . It is easy to see that  $(A_{\Gamma, \bar{\gamma}})_{\bar{\gamma} \in \Gamma/Y}$  gives rise to a Zariski open covering of  $A_{\Gamma}$ .

**PROPOSITION 2.3.** *Let  $\Gamma$  be as in 2.2 with the additional condition  $\{0\} \in \Gamma$ . Then  $A_{\Gamma}$  contains  $G$  as an open algebraic subspace canonically.*

**PROOF.** Let  $\iota_1$  (resp.  $\iota_2$ ) denote the subdivision  $\Xi \sqcap \Gamma \rightarrow \Xi$  (resp.  $\Xi \sqcap \Gamma \rightarrow \Gamma$ ) as in diagram (2.5). Under our assumption we have  $\{0\} \in \Xi \cap (\Xi \sqcap \Gamma) \cap \Gamma$ , hence



all the  $Y$ -translates  $\{y\}$  of  $\{0\}$  lie in  $\Xi \cap (\Xi \cap \Gamma) \cap \Gamma$ . Let

$$\begin{aligned} Q_\Xi &= P_\Xi - \bigcup_{y \in Y} P_{\{y\}}, & \mathcal{Z}_\Xi &= \varinjlim_n (Q_\Xi \times_S S_n)/Y, \\ Q_{\Xi \cap \Gamma} &= P_{\Xi \cap \Gamma} - \bigcup_{y \in Y} P_{\{y\}}, & \mathcal{Z}_{\Xi \cap \Gamma} &= \varinjlim_n (Q_{\Xi \cap \Gamma} \times_S S_n)/Y, \\ Q_\Gamma &= P_\Gamma - \bigcup_{y \in Y} P_{\{y\}}, & \mathcal{Z}_\Gamma &= \varinjlim_n (Q_\Gamma \times_S S_n)/Y, \end{aligned}$$

it is easy to see that  $Q_\Xi$  (resp.  $Q_{\Xi \cap \Gamma}$ , resp.  $Q_\Gamma$ ) is a reduced closed subscheme of  $P_\Xi$  (resp.  $P_{\Xi \cap \Gamma}$ , resp.  $P_\Gamma$ ), and  $\mathcal{Z}_\Xi$  (resp.  $\mathcal{Z}_{\Xi \cap \Gamma}$ , resp.  $\mathcal{Z}_\Gamma$ ) is a closed formal subscheme of  $\mathcal{A}_\Xi$  (resp.  $\mathcal{A}_{\Xi \cap \Gamma}$ , resp.  $\mathcal{A}_\Gamma$ ). By Grothendieck existence theorem [Knu71, Chap. 5, Sec. 6],  $\mathcal{Z}_\Xi$  (resp.  $\mathcal{Z}_{\Xi \cap \Gamma}$ , resp.  $\mathcal{Z}_\Gamma$ ) is the formal completion of a reduced closed algebraic subspace  $Z_\Xi$  (resp.  $Z_{\Xi \cap \Gamma}$ , resp.  $Z_\Gamma$ ). The algebraic space  $Z_\Xi$  (resp.  $Z_{\Xi \cap \Gamma}$ , resp.  $Z_\Gamma$ ) is supported on the special fibre, hence has the same support as  $\mathcal{Z}_\Xi$  (resp.  $\mathcal{Z}_{\Xi \cap \Gamma}$ , resp.  $\mathcal{Z}_\Gamma$ ).

Now the morphisms  $M(\iota_1)$  and  $M(\iota_2)$  restrict to the identity on  $A_K$  on the generic fibres, and restrict to  $\mathcal{F}(\iota_1)_0$  and  $\mathcal{F}(\iota_2)_0$  on the special fibres respectively. It is easy to see that  $\mathcal{F}(\iota_1)_0$  (resp.  $\mathcal{F}(\iota_2)_0$ ) restricts to the identity over the open subscheme  $(\cup_{y \in Y} P_{\{y\}}) \times_S s$  of  $P_{\Xi,0}$  (resp.  $P_{\Gamma,0}$ ). It follows that both  $M(\iota_1)$  and  $M(\iota_2)$  restrict to isomorphisms over  $A_{\Xi \cap \Gamma} - Z_{\Xi \cap \Gamma}$ , hence  $A_\Gamma - Z_\Gamma = A_{\Xi \cap \Gamma} - Z_{\Xi \cap \Gamma} = A_\Xi - Z_\Xi = G$ .  $\square$

**PROPOSITION 2.4.** *Let  $\iota : \Sigma \rightarrow \Gamma$  be a map of admissible polytope decompositions. Then for any subset  $\bar{\Sigma}_0$  of  $\bar{\Sigma} \cap \bar{\Gamma}$ , which is stable under taking faces,  $M(\iota)$  restricts to an isomorphism on  $A_{\Sigma, \bar{\Sigma}_0}$ .*

**PROOF.** It is enough to consider the case  $\bar{\Sigma}_0 = \bar{\Sigma} \cap \bar{\Gamma}$ . Firstly,  $M(\iota)_K$  is an isomorphism. Since  $M(\iota)_n$  coincides with the morphism  $P_{\Sigma, n}/Y \rightarrow P_{\Gamma, n}/Y$ ,  $M(\iota)^\wedge$  restricts to an isomorphism on  $\bigcup_{\sigma \in \Sigma \cap \Gamma} O_\sigma$ , hence  $M(\iota)$  is étale and radicial on  $\bigcup_{\sigma \in \Sigma \cap \Gamma} O_\sigma$ . Then the result follows from [Gro67, 17.9.1].  $\square$

Now we describe some examples of models associated to certain polytope decompositions which are going to be used in later sections.

**EXAMPLE 2.1.** Choose suitable basis for  $X$  and  $Y$  such that the pairing  $X \times Y \rightarrow \mathbb{Z}$  is given by a diagonal matrix  $\text{diag}(n_1, \dots, n_d)$ <sup>1</sup>. Then  $Y$  sits in  $E$  as the lattice

$$\bigoplus_i n_i \mathbb{Z} e_i$$

via the embedding  $Y \hookrightarrow E$ , where  $e_1, \dots, e_d$  is the corresponding basis of  $E$ . Consider the admissible polytope decomposition  $\Sigma_{\square^d}$  given by all the  $Y$ -translates of the faces of  $\square^d$ , where  $\square^d$  is the  $d$ -cube with vertices  $a_1 e_1 + \dots + a_d e_d$ ,  $a_i \in \{0, n_i\}$ , we get an algebraic space associated to  $\Sigma_{\square^d}$ , and denote it by  $A_Y$ .

<sup>1</sup>Note this does not imply the corresponding matrix for the pairing  $\langle, \rangle : X \times Y \rightarrow \mathbb{G}_{m, K}$  is diagonal. Actually if we can make the matrix diagonal under some choice of basis, then  $A_K$  is isomorphic to a product of some Tate curves.

Now we construct a model for  $A_K \times A_K$ . Consider the lattice

$$Y \times Y = (\oplus_i n_i \mathbb{Z} e_i) \oplus (\oplus_i n_i \mathbb{Z} e_i)$$

in  $E \times E$ , let  $\square^{2d}$  be the  $2d$ -cube with vertices

$$(a_1 e_1 + \cdots + a_d e_d, b_1 e_1 + \cdots + b_d e_d)$$

with  $a_i, b_i \in \{0, n_i\}$  for all  $i$ . The hyperplanes  $a_i = b_i$  divide  $\square^{2d}$  into  $2^d$  polytopes

$$\square_{\underline{u}} = \square^{2d} \cap \left( \bigcap_i H_{u_i} \right)$$

with  $\underline{u} = (u_1, \dots, u_d) \in \{0, 1\}^d$  and

$$H_{u_i} = \begin{cases} a_i \geq b_i & \text{if } u_i = 0; \\ a_i \leq b_i & \text{if } u_i = 1. \end{cases}$$

Take the  $Y \times Y$ -admissible polytope decomposition of  $E \times E$  given by all the  $Y \times Y$ -translates of the faces of the  $\square_{\underline{u}}$ 's, we denote the associated model of  $A_K \times A_K$  by  $(A_Y \times A_Y)_{\square}$ . Apparently we have a morphism of algebraic spaces

$$(A_Y \times A_Y)_{\square} \rightarrow A_Y \times A_Y$$

over  $S$  coming from the above subdivision of  $Y \times Y$ -admissible polytope decompositions. Under the linear map  $E \times E \rightarrow E, (a, b) \mapsto -a + b$ , any  $Y \times Y$ -translate of a  $\square_{\underline{u}}$  lies in some  $Y$ -translate of  $\square^d$ . It follows that we have a morphism

$$(A_Y \times A_Y)_{\square} \xrightarrow{m_-} A_Y$$

of algebraic spaces.

Similarly, we divide the polytope  $\square^{2d}$  into  $2^d$  polytopes  $\square_{\underline{u}}$  by the hyperplanes  $a_i + b_i = 1$ , hence get a proper model  $(A_Y \times A_Y)_{\square}$  equipped with the following morphisms

$$\begin{aligned} (A_Y \times A_Y)_{\square} &\rightarrow A_Y \times A_Y \\ (A_Y \times A_Y)_{\square} &\xrightarrow{m_+} A_Y \end{aligned}$$

where the corresponding map  $E \times E \rightarrow E$  for the second morphism is given by  $(a, b) \mapsto a + b$ .

**PROPOSITION 2.5.** *The open immersion  $G \times_S A_Y \hookrightarrow A_Y \times_S A_Y$  factors canonically as*

$$\begin{array}{ccc} G \times_S A_Y & \xhookrightarrow{\quad} & A_Y \times_S A_Y \\ & \searrow & \nearrow \\ & (A_Y \times A_Y)_{\square} & \end{array}$$

and

$$\begin{array}{ccc} G \times_S A_Y & \xhookrightarrow{\quad} & A_Y \times_S A_Y \\ & \searrow & \nearrow \\ & (A_Y \times A_Y)_{\square} & \end{array}$$

PROOF. The proof is similar to the proof of theorem 2.3 by considering all the  $Y \times Y$ -translates of  $\{0\} \times \tau$ 's for all faces  $\tau$  of  $\square^d$ , instead of considering all  $Y$ -translates of  $\{0\}$ .  $\square$

We define a morphism  $\rho$  as the composition of

$$G \times_S A_Y \hookrightarrow (A_Y \times A_Y)_{\square} \xrightarrow{m_+} A_Y$$

then the morphism  $\rho$  fits into the following commutative diagram

$$(2.8) \quad \begin{array}{ccc} G \times G & \xrightarrow{m_G} & G \\ \downarrow & & \downarrow \\ G \times A_Y & \xrightarrow{\rho} & A_Y \\ & \searrow & \nearrow \\ & (A_Y \times A_Y)_{\square} & \end{array}$$

where  $m_G$  denotes the group law on  $G$ . The diagram suggests that we may expect  $\rho$  to be a group action. This is indeed the case, and we will prove this after example 2.2.

EXAMPLE 2.2. Let the notations be as in (2.1). Now we construct some models for  $A_K \times A_K \times A_K$ . Consider the lattice

$$Y \times Y \times Y = (\oplus_i n_i \mathbb{Z} e_i) \oplus (\oplus_i n_i \mathbb{Z} e_i) \oplus (\oplus_i n_i \mathbb{Z} e_i)$$

in  $E \times E \times E$ , let  $\square^{3d}$  be the  $3d$ -cube with vertices

$$(a_1 e_1 + \cdots + a_d e_d, b_1 e_1 + \cdots + b_d e_d, c_1 e_1 + \cdots + c_d e_d)$$

with  $a_i, b_i, c_i \in \{0, n_i\}$ . The  $Y \times Y \times Y$ -translates of the faces of  $\square^{3d}$  give rise to a  $Y \times Y \times Y$ -admissible polytope decomposition of  $E \times E \times E$ , and we denote it by  $\Sigma_{\square^{3d}}$ . The associated model to  $\Sigma_{\square^{3d}}$  is just  $A_{\square^d} \times A_{\square^d} \times A_{\square^d}$ .

By cutting  $\square^{3d}$  with the hyperplanes

$$a_i + b_i + c_i = 1, a_i + b_i + c_i = 2, a_i + b_i = 1, b_i + c_i = 1$$

with  $i$  varying from 1 to  $d$ , we get a subdivision of  $\square^{3d}$ . Taking the  $Y \times Y \times Y$ -translates of this subdivision, we get a  $Y \times Y \times Y$ -admissible polytope decomposition of  $E \times E \times E$  and we denote it by  $\Sigma_{\boxtimes^{3d}}$ . We denote the model associated to  $\Sigma_{\boxtimes^{3d}}$  by  $(A_Y \times A_Y \times A_Y)_{\boxtimes^{3d}}$ . The decomposition  $\Sigma_{\boxtimes^{3d}}$  is clearly a subdivision of  $\Sigma_{\square^{3d}}$ , whence a canonical morphism  $(A_Y \times A_Y \times A_Y)_{\boxtimes^{3d}} \rightarrow A_Y \times A_Y \times A_Y$ .

The polytope decompositions  $\Sigma_{\boxtimes^{3d}}$ ,  $\Sigma_{\square^{2d}}$  and  $\Sigma_{\square^d}$  are compatible with the commutativity of the diagram

$$\begin{array}{ccc} E \times E \times E & \xrightarrow{+E \times 1_E} & E \times E \\ \downarrow 1_E \times +E & & \downarrow +E \\ E \times E & \xrightarrow{+E} & E \end{array}$$

where  $+_E$  denotes the addition of  $E$ . Hence we get a commutative diagram

$$\begin{array}{ccc} (A_Y \times A_Y \times A_Y)_{\boxtimes^{3d}} & \xrightarrow{m_{+,12}} & (A_Y \times A_Y)_{\boxtimes} \\ m_{+,23} \downarrow & & \downarrow m_+ \\ (A_Y \times A_Y)_{\boxtimes} & \xrightarrow{m_+} & A_Y \end{array}$$

Similar as in the previous proposition, we have a canonical factorisation

$$\begin{array}{ccc} G \times_S G \times_S A_Y & \hookrightarrow & A_Y \times_S A_Y \times_S A_Y \\ & \searrow & \nearrow \\ & (A_Y \times A_Y \times A_Y)_{\boxtimes^{3d}} & \end{array}$$

with the two hook arrows open immersions. Furthermore, the open immersions  $G \times_S G \times_S A_Y \hookrightarrow (A_Y \times A_Y \times A_Y)_{\boxtimes^{3d}}$  and  $G \times_S A_Y \hookrightarrow (A_Y \times A_Y)_{\boxtimes^{2d}}$  are compatible with the “partial addition” morphisms  $m_{+,12}$  and  $m_{+,23}$ , in the sense that they fit into the following commutative diagram

(2.9)

$$\begin{array}{ccccc} G \times_S G \times_S A_Y & \xrightarrow{m_G \times 1_{A_Y}} & G \times_S A_Y & & \\ \downarrow 1_G \times \rho & \searrow & \downarrow & \searrow & \\ & (A_Y \times A_Y \times A_Y)_{\boxtimes^{3d}} & \xrightarrow{m_{+,12}} & (A_Y \times A_Y)_{\boxtimes} & \\ & \downarrow m_{+,23} & & \downarrow m_+ & \\ G \times_S A_Y & \searrow & (A_Y \times A_Y)_{\boxtimes} & \xrightarrow{m_+} & A_Y \end{array}$$

PROPOSITION 2.6. *The morphism  $\rho : G \times_S A_Y \rightarrow A_Y$  defines a  $G$ -action on  $A_Y$ .*

PROOF. The compatibility axiom for group action follows from the commutativity of the diagram (2.9). We are left to check for the role of the identity section  $e_G : S \rightarrow G$ , i.e. to check the commutativity of the following diagram

$$\begin{array}{ccc} G \times A_Y & \xrightarrow{\rho} & A_Y \\ e_G \times 1_{A_Y} \uparrow & \nearrow & \\ A_Y & & \end{array}$$

But  $\rho \circ (e \times 1_{A_Y}) = 1_{A_Y}$  formally and  $A_Y$  is proper over  $S$ , hence  $\rho \circ (e \times 1_{A_Y}) = 1_{A_Y}$ .  $\square$

REMARK 2.1. For the model  $A_\Sigma$  associated to a  $Y$ -admissible polytope decomposition  $\Sigma$  with  $0 \in \Sigma$ , we could ask if the translation action on  $G$  itself extends to

$A_\Sigma$ . The proposition 2.6 offers an affirmative answer for the special case  $A_Y$ . For 1-dimensional case, this is known, see [DR73]. For the case that the decomposition  $\Sigma$  offers an ample line bundle, the answer is mentioned to be yes in [Ale02, 5.7.1], but I couldn't find the arguments for proving this there. Probably the answer yes is well-known to experts, here we give an explicit proof for the special case  $A_Y$  for our purpose.

### 3. A canonical logarithmic structure on $A_\Sigma$

First we investigate the algebraic space  $A_\Sigma$  in more detail.

LEMMA 3.1. *Let  $A_\sigma$  be as in 2.1 (b), let  $\hat{A}_\sigma$  be the  $\pi$ -adic completion of  $A_\sigma$ ,  $\tilde{P}_\sigma = \text{Spec} \hat{A}_\sigma$ , and  $\mathcal{P}_\sigma = \text{Spf} \hat{A}_\sigma$ . Then we have:*

- (i) *The formal scheme  $\mathcal{P}_\sigma$  is normal and Cohen-Macaulay, the ring  $\hat{A}_\sigma$  is normal and Cohen-Macaulay;*
- (ii) *If  $P_\sigma$  is regular, so are  $\tilde{P}_\sigma$  and  $\mathcal{P}_\sigma$ .*

PROOF. For any point  $x \in \mathcal{P}_\sigma$ , we have a sequence of local homomorphisms of noetherian local rings

$$\mathcal{O}_{P_\sigma, x} \xrightarrow{\delta} \mathcal{O}_{\tilde{P}_\sigma, x} \xrightarrow{\lambda} \mathcal{O}_{\mathcal{P}_\sigma, x} \xrightarrow{\mu} \hat{\mathcal{O}}_{\tilde{P}_\sigma, x} = \hat{\mathcal{O}}_{P_\sigma, x}$$

where the completions means the  $\pi$ -adic completion. Both  $\lambda$  and  $\mu$  are faithful flat, see [GM71, 3.1.2]. The composition  $\mu \circ \lambda \circ \delta$  is just the canonical homomorphism from a ring to its completion.

As a complete discrete valuation ring,  $R$  is an excellent ring, see [Gro64, 7.8.3 (iii)]. Since  $A_\sigma$  is a finitely generated  $R$ -algebra, it is excellent, so is  $\mathcal{O}_{P_\sigma, x}$ , see [Gro64, 7.8.3 (ii)]. Then by [Gro64, 7.8.3 (v)], we have  $\hat{\mathcal{O}}_{P_\sigma, x}$  is normal and Cohen-Macaulay, and it is also regular if  $A_\sigma$  is regular at  $x$ . Since  $\mu$  and  $\lambda$  are faithful flat, we have that  $\mathcal{O}_{\mathcal{P}_\sigma, x}$  and  $\mathcal{O}_{\tilde{P}_\sigma, x}$  are normal and Cohen-Macaulay by [Mat80, 21.E], and they are also regular if  $\mathcal{O}_{P_\sigma, x}$  is. Hence (i) and (ii) follow.  $\square$

COROLLARY 3.1. *The formal scheme  $\mathcal{A}_\Sigma$  is normal and Cohen-Macaulay. It is also regular if  $\Sigma$  is regular.*

PROOF. Since normality, regularity and being Cohen-Macaulay are all local properties for étale topology, we are reduced to check for  $\text{Spf} A_\sigma$  for  $\sigma \in \Sigma$ , which follows from the previous lemma.  $\square$

PROPOSITION 3.1. *The algebraic space  $A_\Sigma$  is normal and Cohen-Macaulay, and it is also regular if  $\Sigma$  is regular.*

PROOF. Since  $(A_\Sigma)_K$  is an abelian variety over  $K$ , it is a regular scheme. We are left to consider the points in  $A_\Sigma \setminus (A_\Sigma)_K$ . For any  $x \in A_\Sigma \setminus (A_\Sigma)_K$ , choose an open affine étale neighborhood  $(U, u)$ , with  $U = \text{Spec} B$  and  $u$  lying over  $x$ , we are left to show for  $u \in U$ .

Let  $\hat{B}$  be the  $\pi$ -adic completion of  $B$ , and  $\mathcal{U} = \text{Spf} \hat{B}$ . We have a canonical étale morphism  $\mathcal{U} \rightarrow \mathcal{A}_\Sigma$  associated to  $U$ . Since  $\mathcal{A}_\Sigma$  is normal and Cohen-Macaulay, so

$\mathcal{U}$  is normal and Cohen-Macaulay, and the ring  $\hat{B}$  is normal and Cohen-Macaulay. Consider the following commutative diagram

$$(3.1) \quad \begin{array}{ccc} B & \xrightarrow{\quad} & \hat{B} \\ & \searrow & \nearrow \\ & (1 + (\pi))^{-1}B & \end{array}$$

Since  $\pi$  is contained in the radical of  $(1 + (\pi))^{-1}B$ , we have that  $(1 + (\pi))^{-1}B$  is normal and Cohen-Macaulay by [Gro64, 7.8.3 (v)]. In particular,  $B$  is normal and Cohen-Macaulay at  $u$ . It follows the algebraic space  $A_\Sigma$  is normal and Cohen-Macaulay. The regularity part can be proven by similar arguments.  $\square$

Now we define a canonical log structure  $M$  (resp.  $\mathcal{M}$ ) on  $A_\Sigma$  (resp.  $\mathcal{A}_\Sigma$ ) by letting

$$(3.2) \quad M(U) = \{f \in \mathcal{O}_{A_\Sigma}(U) \mid f \in (\mathcal{O}_{A_\Sigma}(U) \otimes_R K)^\times\}$$

$$(3.3) \quad (\text{resp. } \mathcal{M}(U) = \{f \in \mathcal{O}_{\mathcal{A}_\Sigma}(U) \mid f \in (\mathcal{O}_{\mathcal{A}_\Sigma}(U) \otimes_R K)^\times\})$$

for any open  $U$  in  $(A_\Sigma)_{\text{ét}}$  (resp.  $(\mathcal{A}_\Sigma)_{\text{ét}}$ ). This makes  $A_\Sigma$  (resp.  $\mathcal{A}_\Sigma$ ) into a log algebraic space (resp. log formal scheme) over the log scheme  $S$  (resp. the log formal scheme  $\mathcal{S}$ ). We have a canonical morphism

$$(3.4) \quad \iota : \mathcal{A}_\Sigma \rightarrow A_\Sigma$$

of ringed spaces. This further gives a morphism

$$(3.5) \quad (\mathcal{A}_\Sigma)_{\text{ét}} \rightarrow (A_\Sigma)_{\text{ét}}$$

of small étale sites. Here for the definition of a morphism between sites, we refer to [Sta14, 7.15.1], for the proof of (3.5) being a morphism of sites, we use [Sta14, 7.15.5] (the category  $(\mathcal{I}_V^u)^{\text{opp}}$  in [Sta14, 7.15.5] is obviously filtered in our case). Hence the pullback functor

$$\iota^{-1} : (\mathcal{A}_\Sigma)_{\text{ét}}^\sim \rightarrow (A_\Sigma)_{\text{ét}}^\sim$$

is exact.

Note that the log structures  $M$  and  $\mathcal{M}$  are defined in a similar way, and actually they are closely related along the morphism  $\iota$ . The structure morphism  $M \rightarrow \mathcal{O}_{A_\Sigma}$  gives rise to

$$\iota^{-1}M \rightarrow \iota^{-1}\mathcal{O}_{A_\Sigma} \rightarrow \mathcal{O}_{\mathcal{A}_\Sigma}$$

and we let  $\beta$  be the composition. It is easy to see that  $\beta$  factors through  $\mathcal{M} \rightarrow \mathcal{O}_{\mathcal{A}_\Sigma}$ , in other words we have the following commutative diagram

$$(3.6) \quad \begin{array}{ccc} \iota^{-1}M & \hookrightarrow & \iota^{-1}\mathcal{O}_{A_\Sigma} \\ \downarrow & \searrow \beta & \downarrow \\ \mathcal{M} & \hookrightarrow & \mathcal{O}_{\mathcal{A}_\Sigma} \end{array}$$

Then by the definition of  $\iota^*M$ , we get a canonical morphism  $\eta$

$$(3.7) \quad \begin{array}{ccc} \beta^{-1}(\mathcal{O}_{\mathcal{A}_\Sigma}^\times) & \hookrightarrow & \iota^{-1}M \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{A}_\Sigma}^\times & \longrightarrow & \iota^*M \\ & \searrow & \swarrow \eta \\ & & \mathcal{M} \end{array}$$

coming out of the universal property of pushout. And  $\eta$  is actually a morphisms of log structures on  $\mathcal{A}_\Sigma$ .

**PROPOSITION 3.2.** *The canonical morphism  $\eta$  is an isomorphism of log structures on  $\mathcal{A}_\Sigma$ .*

**PROOF.** The morphism  $\eta$  fits naturally into the following commutative diagram

$$(3.8) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}_{\mathcal{A}_\Sigma}^\times & \longrightarrow & \iota^*M & \longrightarrow & \overline{\iota^*M} \longrightarrow 1 \\ & & \parallel & & \downarrow \eta & & \downarrow \bar{\eta} \\ 1 & \longrightarrow & \mathcal{O}_{\mathcal{A}_\Sigma}^\times & \longrightarrow & \mathcal{M} & \longrightarrow & \overline{\mathcal{M}} \longrightarrow 1 \end{array}$$

with exact rows. Note here the short sequence  $1 \rightarrow T' \xrightarrow{u} T \xrightarrow{v} T'' \rightarrow 1$  being exact means that  $T'$  is a subsheaf of groups of  $T$  such that  $T''$  is the associated quotient in the category of sheaves of monoids which is not an abelian category. And this notion of exactness is just for our purpose here, not a standard terminology. To prove  $\eta$  is an isomorphism, it is enough to show that  $\bar{\eta}$  is an isomorphism. By [Kat96, 3.3], we have a canonical isomorphism  $\overline{\iota^*M} \cong \iota^{-1}(\overline{M})$ . So we are left to prove that  $\iota^{-1}(\overline{M})$  is isomorphic to  $\overline{\mathcal{M}}$  under  $\bar{\eta}$ , and this follows from the following lemma.  $\square$

**LEMMA 3.2.** *Let  $B$  be a noetherian normal domain,  $I$  an ideal in  $B$ ,  $\hat{B}$  the  $I$ -adic completion of  $B$ . Let  $f$  be an element in  $\hat{B}$  which is not a zero-divisor, such that  $V(f) \subseteq V(I\hat{B}) = V(I)$ . Then we can find a Zariski covering  $\{U_j = \text{Spec} B_j\}_{j \in J}$  of the scheme  $U = \text{Spec} B$ , such that  $f = g_j u_j$  on  $\hat{B}_j$  for some  $g_j \in B_j$  and  $u_j \in \hat{B}_j^\times$ .*

**PROOF.** First of all, the element  $f$  defines an effective principal Cartier divisor, hence a closed subscheme of  $\text{Spec} \hat{B}$ , see [Gro67, 21.2.12], which further gives a codim 1 cycle  $D = \sum_i n_i D_i$ . Now we regard  $D$  as a codim 1 cycle on  $\text{Spec} B$ , and by [Gro67, 21.7.2] we get a closed subscheme  $Y(D)$  of  $U$ . Let  $I_D$  be the sheaf of ideals of  $Y(D)$ . Set-theoretically,  $Y(D)$  is contained in  $V(I)$ , hence we have  $I^r \subseteq I_D$  for some big integer  $r$ ,  $I_D$  is an open ideal. It follows that  $B/I_D \cong \hat{B}/I_D \hat{B}$ , and  $I_D \hat{B}$  gives rise to the cycle  $D$  too. Again by [Gro67, 21.7.2], we get  $(f) = I_D \hat{B}$ .

Consider the following commutative diagram of  $I$ -adic rings

$$(3.9) \quad \begin{array}{ccc} B & \xrightarrow{\quad} & \hat{B} \\ & \searrow \quad \nearrow & \\ & B' & \end{array}$$

where  $B'$  denotes the ring  $(1 + I)^{-1}B$  which is an Zariski ring. Since  $(f) = I_D \hat{B} = (I_D B') \hat{B}$ , we have that  $I_D B'$  is principal by [Mat80, 24.E (i)]. It is obvious that  $I_D$  is principal on  $U \setminus V(I)$ , we only need to show that  $I_D$  is principal at  $B_{\mathfrak{p}}$  for all prime ideals  $\mathfrak{p} \in V(I)$ . But for  $\mathfrak{p} \supseteq I$  we have  $\mathfrak{p} \cap (1 + I) = \emptyset$ , hence  $I_D B'$  being principal implies that  $I_D B_{\mathfrak{p}}$  is principal.  $\square$

From the above proposition, we could describe explicitly the pullback of the log structure  $M$  on  $A_{\Sigma}$  to  $S_n$ .

**COROLLARY 3.2.** *The pullback of the log structure  $M$  on  $A_{\Sigma}$  to  $S_n$  admits étale local charts  $C(\sigma)^{\vee} \cap \mathbb{X}' \longrightarrow R[C(\sigma)^{\vee} \cup \mathbb{X}']/(\pi^{n+1})$ .*

#### 4. Construction of logarithmic abelian varieties

In this section, we are going to construct the log abelian variety over  $S$  extending  $A_K$ . We follow the paper [KKN08a] in particular section 5 closely.

**4.1.** In this subsection, we review some constructions from log geometry. Here we only work over  $S$ , but most constructions work over general bases. We define the log multiplicative group  $\mathbb{G}_{m,\log}$  to be the sheaf of abelian group on  $(\text{fs}/S)$  defined by

$$\mathbb{G}_{m,\log}(U) := M_U^{\text{gp}}(U)$$

for any  $U \in (\text{fs}/S)$ , here  $M_U$  denotes the log structure on  $U$ . It is easy to see that the multiplicative group  $\mathbb{G}_m$  sits in  $\mathbb{G}_{m,\log}$  canonically, so we have a short exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_{m,\log} \rightarrow \mathbb{G}_{m,\log}/\mathbb{G}_m \rightarrow 0$$

We define the log torus  $T_{\log}$  associated to  $X$  to be  $\mathcal{H}om(X, \mathbb{G}_{m,\log})$ , and we have a canonical short exact sequence

$$0 \rightarrow T \rightarrow T_{\log} \rightarrow \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m) \rightarrow 0$$

Recall we have a bilinear form  $\langle, \rangle: X \times Y \rightarrow K^{\times}$ , see (1.4), this gives rise to a bilinear form

$$(4.1) \quad \langle, \rangle: X \times Y \rightarrow \mathbb{G}_{m,\log},$$

hence a bilinear form

$$(4.2) \quad \langle, \rangle: X \times Y \rightarrow \mathbb{G}_{m,\log}/\mathbb{G}_m.$$

Here by abuse of notation, we use  $\langle, \rangle$  to denote all the three pairings, this shouldn't lead to any confusion since their meaning could be read out from the context. We



define a subgroup sheaf  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$  of the sheaf  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$  by

$$\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}(U) := \left\{ \varphi \in \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)(U) \mid \begin{array}{l} \forall u \in U, x \in X, \exists y, y' \in Y, \\ \text{s.t. } \langle x, y \rangle_{\bar{u}} \mid \varphi(x)_{\bar{u}} \mid \langle x, y' \rangle_{\bar{u}} \end{array} \right\}$$

for  $U \in (\text{fs}/S)$ , here  $\bar{u}$  denotes a geometric point above  $u$ . The pairing  $\langle, \rangle$  induces a homomorphism  $Y \rightarrow \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ , which admits the following factorization

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m) \\ & \searrow & \nearrow \\ & \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)} & \end{array}$$

Then we define a subgroup sheaf  $T_{\log}^{(Y)}$  of the sheaf  $T_{\log}$  as the inverse image of  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$  along the homomorphism  $T_{\log} \rightarrow \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$ , whence a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T & \longrightarrow & T_{\log}^{(Y)} & \longrightarrow & \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T & \longrightarrow & T_{\log} & \longrightarrow & \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m) \longrightarrow 0 \end{array}$$

with exact rows.

Now for a given polytope decomposition  $\Sigma$  of  $E$ , we define a subsheaf

$$\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$$

of the sheaf

$$\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

by

$$\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}(U) := \left\{ \varphi \in \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)(U) \mid \begin{array}{l} \forall u \in U, \exists \sigma \in \Sigma, \text{ s.t. } \forall (\mu, x) \in C(\sigma)^\vee, \\ \mu \cdot \varphi(x)_{\bar{u}} \in (M_U/\mathcal{O}^\times)_{\bar{u}} \end{array} \right\}$$

Then we define a subsheaf  $T_{\log}^{(\Sigma)}$  of the sheaf  $T_{\log}$  as the inverse image of

$$\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$$

along the homomorphism

$$T_{\log} \rightarrow \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$$

, whence a commutative diagram

$$\begin{array}{ccccccc}
& T & \longrightarrow & T_{\log}^{(\Sigma)} & \longrightarrow & \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)} & \\
& \parallel & & \downarrow & & \downarrow & \\
0 & \longrightarrow & T & \longrightarrow & T_{\log} & \longrightarrow & \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m) \longrightarrow 0
\end{array}$$

with the second row exact and the first row making  $T_{\log}^{(\Sigma)}$  a  $T$ -torsor over

$$\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$$

- REMARK 4.1. (i) The canonical inclusion  $T \hookrightarrow T_{\log}^{(Y)}$  restricts to an isomorphism on  $K$ , whilst the canonical inclusion  $T \hookrightarrow T_{\log}$  doesn't. The subgroup sheaf  $T_{\log}^{(Y)}$  cuts the very part related to the pairing  $\langle, \rangle$  out of  $T_{\log}$ .
- (ii) The sheaf  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$  is actually a subsheaf of the sheaf  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$ , hence the sheaf  $T_{\log}^{(\Sigma)}$  is a subsheaf of the sheaf  $T_{\log}^{(Y)}$ . But unlike  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(Y)}$  and  $T_{\log}^{(Y)}$ , the subsheaves  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$  and  $T_{\log}^{(\Sigma)}$  are in general not subgroup sheaves of  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)$  and  $T_{\log}$  respectively.

There are many possible polytope decompositions of  $E$ , whence many subsheaves of  $T_{\log}^{(Y)}$ . We would like to know the relation between  $T_{\log}^{(\Sigma)}$  and  $T_{\log}^{(\Sigma')}$  for any two polytope decompositions. We would like to know if it is possible to get  $T_{\log}^{(Y)}$  from the union of  $T_{\log}^{(\Sigma)}$  with  $\Sigma$  varying in a family of polytope decompositions of  $E$ . We could even ask the representability of  $T_{\log}^{(\Sigma)}$  for certain  $\Sigma$ . The followings answer the questions.

PROPOSITION 4.1. *For any two polytope decompositions  $\Sigma$  and  $\Sigma'$  of  $E$ , we have*

$$\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)} \cap \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma')} = \mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma \cap \Sigma')}$$

whence  $T_{\log}^{(\Sigma)} \cap T_{\log}^{(\Sigma')} = T_{\log}^{(\Sigma \cap \Sigma')}$ .

PROOF. Since for  $\sigma \in \Sigma$  and  $\sigma' \in \Sigma'$ , we have  $C(\sigma)^\vee + C(\sigma')^\vee = C(\sigma \cap \sigma')^\vee$ . Then the results follows directly from the definition of  $\mathcal{H}om(X, \mathbb{G}_{m,\log}/\mathbb{G}_m)^{(\Sigma)}$  and  $T_{\log}^{(\Sigma)}$ .  $\square$

EXAMPLE 4.1. Let the notations be as in 2.1 and 2.2. For  $m$  a positive integer, let  $\square_m^d$  be the  $d$ -cube with vertices  $(a_1, \dots, a_d)$ ,  $a_i \in \{0, mn_i\}$ , let  $\Sigma_{\square_m^d}$  be the  $mY$ -admissible polytope decomposition given by the  $mY$ -translates of  $\square_m^d$ . Then we have

$$\bigcup_{m \geq 1, a \in \frac{1}{2}Y} T_{\log}^{(a + \Sigma_{\square_m^d})} = T_{\log}^{(Y)}$$

as sheaves, see [KKN08b, 3.5.4] for the proof. With the help of [KKN08b, 3.5.4], one could also construct other examples.

**PROPOSITION 4.2.** *Let  $\Sigma$  be an  $H$ -admissible polytope decomposition of  $E$  for a cofinite subgroup  $H$  of  $Y$ , then the sheaf  $T_{\log}^{\Sigma}$  is represented by the log  $S$ -scheme  $P_{\Sigma}$  endowed with log structure coming from the monoids  $(C(\sigma)^{\vee} \cap \mathbb{X}')_{\sigma \in \Sigma}$ .*

**PROOF.** See [KKN08b, 3.5.3, 3.5.4].  $\square$

**COROLLARY 4.1.** *Let  $H$  be a cofinite subgroup of  $Y$ ,  $\Sigma$  be an  $H$ -admissible polytope decomposition of  $E$ . Then the pullback of  $A_{H,\Sigma}$  to  $(\text{fs}/S)'$  coincides with the sheaf  $T_{\log}^{(\Sigma)}/H$ .*

**PROOF.** This follows from the proposition 4.2 and the corollary 3.2.  $\square$

**4.2.** Let  $H$  be a cofinite subgroup of  $Y$ ,  $\Sigma$  be an  $H$ -admissible polytope decomposition of  $E$ , we regard the algebraic space  $A_{H,\Sigma}$  as a log algebraic space with respect to the canonical log structure define in (3.2).

Consider the sheaf of sets on  $(\text{fs}/S)$

$$(4.3) \quad A = \left( \coprod_{(H,\Sigma)} A_{H,\Sigma} \right) / \sim$$

where  $(H,\Sigma)$  runs over the pairs with  $H$  a cofinite subgroup of  $Y$  and  $\Sigma$  an  $H$ -admissible polytope decomposition of  $E$ , and  $\sim$  is the equivalence relation in the category of sheaves on  $(\text{fs}/S)$  generated by the following two equivalences:

- (a) For any two pairs  $(H,\Sigma)$  and  $(H',\Sigma')$  such that  $H'$  is a subgroup of  $H$  and  $\Sigma'$  is a subdivision of  $\Sigma$ , we have a canonical morphism

$$A_{H',\Sigma'} \rightarrow A_{H,\Sigma}$$

in  $(\text{fs}/S)$ . Any element of  $A_{H',\Sigma'}(U)$  for  $U \in (\text{fs}/S)$  is equivalent to its image in  $A_{H,\Sigma}(U)$ .

- (b) For any pair  $(H,\Sigma)$  and any  $a \in Y/H$ , we have a morphism  $A_{H,\Sigma} \rightarrow A_{H,a+\Sigma}$  of formal schemes over  $\mathcal{S}$  induced by the multiplication by the element  $\langle \cdot, a \rangle \in T_{\log}$ , hence a morphism  $A_{H,\Sigma} \rightarrow A_{H,a+\Sigma}$  of  $S$ -spaces. Any element of  $A_{H,\Sigma}(U)$  for  $U \in (\text{fs}/S)$  is equivalent to its image in  $A_{H,a+\Sigma}(U)$ .

The main results of this subsection are the following two theorems, which correspond to [KKN08a, 1.7, 4.7].

**THEOREM 4.1.** (a) *The pullback of  $A$  to  $(\text{fs}/K)$  coincides with  $A_K$ . The pullback of  $A$  to  $(\text{fs}/S)'$  coincides with  $T_{\log}^{(Y)}/Y$ .*  
 (b) *There exists a unique group law on  $A$  whose pullback to  $(\text{fs}/K)$  coincides with the group law of  $A_K$ , and whose pullback to  $(\text{fs}/S)'$  coincides with the group law of  $T_{\log}^{(Y)}/Y$ .*

(c) The canonical morphism  $A_{Y,\Sigma} \rightarrow A$  fits into the following Cartesian diagram

$$\begin{array}{ccc} A_{Y,\Sigma} & \xrightarrow{\beta_{Y,\Sigma}} & T_{\log}^{(\Sigma)}/(Y \cdot T) \\ \downarrow & & \downarrow \\ A & \xrightarrow{\beta} & T_{\log}^{(Y)}/(Y \cdot T) \end{array}$$

In other words, the morphism  $A_{Y,\Sigma} \rightarrow A$  is injective, and as a subsheaf of  $A$ , the sheaf  $A_{Y,\Sigma}$  coincides with the inverse image of  $T_{\log}^{(\Sigma)}/(Y \cdot T)$  along  $\beta$ .

(d) With the group law specified in (b),  $A$  fits into a short exact sequence  $0 \rightarrow G \xrightarrow{\alpha} A \xrightarrow{\beta} T_{\log}^{(Y)}/(Y \cdot T) \rightarrow 0$ , where  $\alpha$  is the composition  $G \hookrightarrow A_Y \hookrightarrow A$ , and  $\beta$  is as in (c).

PROOF. The pullback of  $A_{H,\Sigma}$  to  $(\text{fs}/K)$  is an étale covering of  $A_K$  with Galois group  $Y/H$ , hence the pullback of  $A$  to  $(\text{fs}/K)$  is just  $(A_{H,\Sigma})_K/(Y/H) = A_K$ . By 4.1 and 4.2, it is easy to see that the pullback of  $A$  to  $(\text{fs}/S)'$  is just  $T_{\log}^{(Y)}$ . This proves part (a).

The composition of the canonical morphisms

$$A \twoheadrightarrow i_* i^* A = i_* i^* (T_{\log}^{(Y)}/Y) \twoheadrightarrow i_* i^* (T_{\log}^{(Y)}/Y \cdot T) = T_{\log}^{(Y)}/Y \cdot T$$

gives a surjective homomorphism  $\beta : A \twoheadrightarrow T_{\log}^{(Y)}/Y \cdot T$ . Similarly we have a surjective morphism  $\beta_{H,\Gamma} : A_{H,\Gamma} \twoheadrightarrow T_{\log}^{(\Gamma)}/H \cdot T$  for any  $H$ -admissible polytope decomposition  $\Gamma$  with  $H$  a cofinite subgroup of  $Y$ . It is easy to see that  $\beta$  and  $\beta_{Y,\Sigma}$  fit into a canonical commutative diagram

$$\begin{array}{ccc} A_{Y,\Sigma} & \twoheadrightarrow & T_{\log}^{(\Sigma)}/Y \cdot T \\ \downarrow & & \downarrow \\ A & \twoheadrightarrow & T_{\log}^{(Y)}/Y \cdot T \end{array}$$

In order to finish the proof of (c), we need to show that:

For any  $U \in (\text{fs}/S)$ , and any  $f \in A(U)$  such that  $\beta(f) \in (T_{\log}^{(\Sigma)}/Y \cdot T)(U)$ , étale locally on  $U$  we can lift  $f$  to a section of  $A_{Y,\Sigma}$ .

Suppose that  $f$  is represented by a section in  $A_{H,\Gamma}(U)$ , and we still call it  $f$ . According to the diagram

$$\begin{array}{ccc} & A_{H,\Gamma \cap \Sigma} & \\ \swarrow & \downarrow \text{étale quotient} & \\ A_{H,\Gamma} & A_{Y,\Gamma \cap \Sigma} & \\ & \downarrow & \\ & A_{Y,\Sigma} & \end{array}$$

it is enough to lift  $f$  to  $A_{H,\Gamma \cap \Sigma}$  étale locally. We have  $T_{\log}^{(\Gamma)} \cap T_{\log}^{(\Sigma)} = T_{\log}^{(\Gamma \cap \Sigma)}$  by 4.1 and  $\beta(f) \in (T_{\log}^{(\Sigma)}/Y \cdot T)(U)$ , hence  $\beta_{H,\Gamma}(f) \in (T_{\log}^{(\Gamma \cap \Sigma)}/H \cdot T)(U)$ . For any  $u \in U$ , take an affine étale neighborhood  $\text{Spec} B$  of  $u$  and étale open  $\text{Spec} C$  (resp.  $\text{Spec} C'$ ) of  $A_{H,\Gamma}$  (resp.  $A_{H,\Gamma \cap \Sigma}$ ) such that

$$f(\text{Spec} B) \subset \text{Spec} C \leftarrow \text{Spec} C'$$

and this corresponds to  $B \xleftarrow{f^\#} C \xrightarrow{\alpha} C'$ . Taking the  $\pi$ -adic completion, we have  $(f^\#)^\wedge$  factors through  $\hat{\alpha}$  by some  $\hat{R}$ -algebra morphism  $\hat{\theta}$ . We have the following commutative diagram (excluding the dotted arrow)

$$(4.4) \quad \begin{array}{ccccc} B & \xrightarrow{\iota_B} & \hat{B} & & \\ & \nwarrow f^\# & \uparrow (f^\#)^\wedge & & \\ & C & \xrightarrow{\iota_C} & \hat{C} & \\ & \swarrow \alpha & \downarrow \hat{\theta} & \swarrow \hat{\alpha} & \\ C' & \xrightarrow{\iota_{C'}} & \hat{C}' & & \end{array}$$

of  $R$ -algebras, where  $\iota_B, \iota_C$ , and  $\iota_{C'}$  are the canonical inclusions for the completions. Since  $\alpha_K$  is an isomorphism, the composition  $\hat{\theta} \cdot \iota_{C'}$  has image in  $B$ , hence defines a  $S$ -morphism through which  $f$  factors. This finishes the proof of part (c).

We postpone the proof of (b) and (d) to the end of this subsection.  $\square$

**THEOREM 4.2.** *The sheaf  $A$  with the group law specified in 4.1 is a log abelian variety over  $S$  extending the abelian variety  $A_K$  over  $K$ .*

We also postpone the proof of 4.2 to the end of this subsection

**DEFINITION 4.1.** In view of 4.2, we call the sheaf of abelian groups  $A$  the **log abelian variety associated to the period lattice  $Y$** .

**REMARK 4.2.** (a) It is interesting to compare  $A$  with  $T_{\log}^{(Y)}/Y$ .

(b) Any cofinite subgroup  $H$  of  $Y$  can be regarded as a period lattice in  $T_K$  canonically, hence we can define the log abelian variety for  $H$  in the same way as for  $Y$  in (4.3).

Before going to the rest part of the proof of 4.1 and the proof of 4.2, let us first give some lemmas needed for the proofs.

**LEMMA 4.1.** *For any  $Y$ -admissible polytope decomposition  $\Sigma$  of  $E$ , the canonical map  $A_\Sigma \rightarrow A$  is injective.*

**PROOF.** By a limit argument, it is sufficient to prove that the map  $A_\Sigma(U) \rightarrow A(U)$  is injective for any fs log scheme  $U$  over  $S$  whose underlying scheme is the affine scheme for a noetherian ring  $B$ . Since  $B \rightarrow (B \otimes_R K) \times \varprojlim_n (B/\pi^{n+1}B)$  is faithfully flat and  $A_\Sigma(\varprojlim_n (B/\pi^{n+1}B)) = \varprojlim_n A_\Sigma(B/\pi^{n+1}B)$  by [Bha14] (bibtext error, Bhatt: Algebraization and Tannaka duality 4.1), the map  $A_\Sigma(R) \rightarrow A_\Sigma(B \otimes_R K) \times \varprojlim_n A_\Sigma(B/\pi^{n+1}B)$  is injective. Since  $A_\Sigma(B \otimes_R K) = A_K(B \otimes_R K)$ , we are

reduced to show the injectivity of  $A_\Sigma(B/\pi^{n+1}B) \rightarrow A(B/\pi^{n+1}B)$  which is clear from the descriptions of the restrictions of  $A_\Sigma$  and  $A$  to  $(\text{fs}/S)'$ .  $\square$

**COROLLARY 4.2.** *Let  $H$  be a cofinite subgroup of  $Y$ , let  $A'$  be the log abelian variety associated to  $H$  as a period lattice. Then we have:*

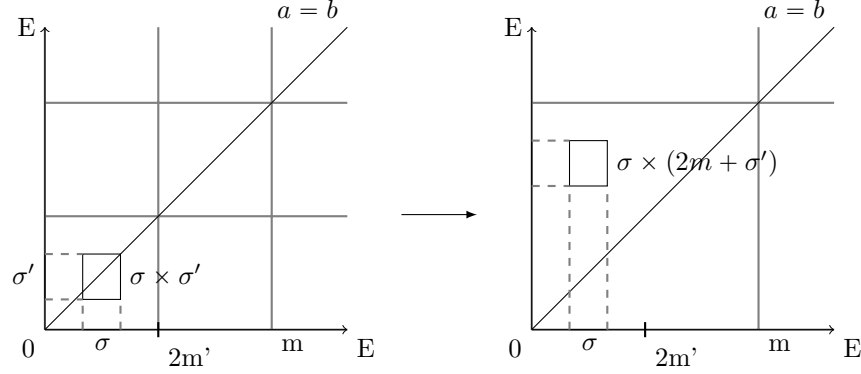
- (a) *The sheaf  $A$  is an étale quotient of  $A'$  under the group action of  $Y/H$ .*
- (b) *For any two  $H$ -admissible polytope decompositions  $\Sigma$  and  $\Gamma$ ,  $A_{H,\Sigma}$  and  $A_{H,\Gamma}$  are two subsheaves of  $A'$  with  $A_{H,\Sigma} \cap A_{H,\Gamma} = A_{H,\Sigma \cap \Gamma}$ . In particular, for  $H = Y$  we have  $A_\Sigma \cap A_\Gamma = A_{\Sigma \cap \Gamma}$  inside  $A$ .*

**PROOF.** Part (a) follows from the definitions of  $A$  and  $A'$ , see (4.3). For (b), it is enough to consider the case  $H = Y$ . We have canonical morphisms  $A_\Sigma \leftarrow A_{\Sigma \cap \Gamma} \rightarrow A_\Gamma$ , which are injective by 4.1. Now for any section  $f \in A_\Sigma \cap A_\Gamma$ , by the equivalent relations in the definition of  $A$ , we can find a  $Y$ -admissible polytope decomposition  $\Delta$  refining both  $\Sigma$  and  $\Gamma$ , such that  $f \in A_\Delta$ . Then we must have that  $\Delta$  also refines  $\Sigma \cap \Gamma$ , hence  $f$  is also a section of  $A_{\Sigma \cap \Gamma}$ . This finishes the proof.  $\square$

**LEMMA 4.2.** *Let  $U$  be an fs log scheme over  $S$ . Let  $f, g \in A(U)$ . Then étale locally on  $U$ , there exist an integer  $m \geq 1$  and sections  $\tilde{f}, \tilde{g}$  of  $A_{mY, \Sigma_{\square_m^d}}$  such that  $f$  (resp.  $g$ ) comes from  $\tilde{f}$  (resp.  $\tilde{g}$ ), and  $(\tilde{f}, \tilde{g})$  belongs to  $(A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}$ . Here  $\square_m^d$  denotes the  $d$ -cube with vertices  $(a_1, \dots, a_d)$ ,  $a_i \in \{0, mn_i\}$ , and  $\Sigma_{\square_m^d}$  is the  $mY$ -admissible polytope decomposition given by the  $mY$ -translates of the faces of  $\square_m^d$ .*

**PROOF.** Suppose that  $f$  (resp.  $g$ ) comes from  $A_{H,\Sigma}$  (resp.  $A_{H',\Sigma'}$ ). Since  $(A_{H,\Sigma,\bar{\sigma}})_{\bar{\sigma} \in \Sigma/H}$  (resp.  $(A_{H',\Sigma',\bar{\sigma}'} )_{\bar{\sigma}' \in \Sigma'/H'}$ ) is an open covering of  $A_{H,\Sigma}$  (resp.  $A_{H',\Sigma'}$ ). We may assume that  $f$  (resp.  $g$ ) comes from  $A_{H,\Sigma,\bar{\sigma}}(U)$  (resp.  $A_{H',\Sigma',\bar{\sigma}'}(U)$ ) for some  $\sigma \in \Sigma$  (resp.  $\sigma' \in \Sigma'$ ).

For  $m$  a positive integer, let  $\square_m^d, \Sigma_{\square_m^d}$  be as in 4.1. We can make  $\sigma, \sigma' \subset -\underline{x} + \square_{2m'}^d$  with  $\underline{x} = (m'n_1, \dots, m'n_d)$  for  $m'$  big enough. According to the equivalent relation (b) in the definition (4.3) of  $A$ , we can replace  $\Sigma, \Sigma'$  by their translates under  $\underline{x}$ , so that  $\sigma, \sigma' \subset \square_{2m'}^d$ . Now let  $m = 4m'$ , and we further replace  $\Sigma'$  by its translate under  $(2m'n_1, \dots, 2m'n_d)$ , then we have  $\sigma, \sigma' \subset \square_m^d$ , and  $\sigma \times \sigma'$  goes into  $\square_m^d$  under the map  $E \times E \rightarrow E, (a, b) \mapsto (-a + b)$ .



So we get the following diagram

$$\begin{array}{ccccc}
 A_{mY, \Sigma, \bar{\sigma}} & \xrightarrow{\alpha} & A_{mY, \Sigma_{\square_m^d}, \square_m^d} & \xleftarrow{\alpha'} & A_{mY, \Sigma', \bar{\sigma}'} \\
 \text{étale} \downarrow & & & & \downarrow \text{étale} \\
 A_{H, \Sigma, \bar{\sigma}} & & & & A_{H', \Sigma', \bar{\sigma}'}
 \end{array}$$

with the following canonical factorization

$$\begin{array}{c}
 A_{mY, \Sigma, \bar{\sigma}} \times A_{mY, \Sigma', \bar{\sigma}'} \xrightarrow{\alpha \times \alpha'} A_{mY, \Sigma_{\square_m^d}, \square_m^d} \times A_{mY, \Sigma_{\square_m^d}, \square_m^d} \longrightarrow A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}} \\
 \searrow \qquad \qquad \qquad \nearrow \\
 (A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}
 \end{array}$$

It follows that, étale locally, we can make  $f$  and  $g$  come from  $\tilde{f}$  and  $\tilde{g}$  in  $A_{mY, \square}$  respectively such that  $(\tilde{f}, \tilde{g})$  belongs to  $(A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}$ .  $\square$

LEMMA 4.3. *Let  $\text{Spec} B$  be an fs log scheme over  $S$  with  $B$  a noetherian ring. Then the canonical map  $A(B) \rightarrow A(B \otimes_R K) \times \varprojlim A(B/\pi^{n+1}B)$  is injective.*

PROOF. Let  $f, g \in A(B)$ , we assume that the image of  $f$  and  $g$  in  $A(B \otimes_R K) \times \varprojlim A(B/\pi^{n+1}B)$  coincide. We want to prove  $f = g$ . By 4.2, étale locally on  $\text{Spec} B$ , there exists  $m \geq 1$  such that  $f$  and  $g$  come from  $\tilde{f}$  and  $\tilde{g}$  in  $A_{mY, \Sigma_{\square_m^d}}$  respectively and such that  $(\tilde{f}, \tilde{g}) \in (A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}$ . Since the images of  $f, g$  in  $T_{\log}^{(Y)}/(T \cdot Y)$  coincide, there exists  $y \in Y$  such that the images of  $\langle \cdot, y \rangle \cdot \tilde{f}$  and  $\tilde{g}$  in  $T_{\log}^{(Y)}/(T \cdot (mY))$  coincide. Note that  $\langle \cdot, y \rangle \cdot \tilde{f}$  is not only a section of  $A_{mY, y + \Sigma_{\square_m^d}}$ , but also a section of  $A_{mY, \Sigma_{\square_m^d}}$  by 4.1 (c), because of its image in  $T_{\log}^{(Y)}/(T \cdot (mY))$  lands in  $T_{\log}^{(\Sigma_{\square_m^d})}/(T \cdot (mY))$ .

Now let  $G'$  be the semiabelian scheme over  $S$  corresponding to the abelian variety  $(A_{mY, \Sigma_{\square_m^d}})_K$ , and we have the following diagram

$$\begin{array}{ccc} G' & \xrightarrow{\text{open}} & A_{mY, \Sigma_{\square_m^d}} \\ \text{quasi-finite étale} \downarrow & & \\ G & & \end{array}$$

Since the image of  $a := m_-(\langle \cdot, y \rangle \tilde{f}, \tilde{g})$  in  $T_{\log}^{(\Sigma_{\square_m^d})}/(T \cdot (mY))$  is the identity of  $T_{\log}^{(Y)}/(T \cdot (mY))$ , the section  $a$  of  $A_{mY, \Sigma_{\square_m^d}}$  actually lies in the open subspace  $G'$ , i.e.  $a \in G'(B)$ . Here  $m_-$  is the morphism defined in 2.1. By assumption, the images of  $a$  in  $G(B \otimes_R K)$  and  $\varprojlim_n G(B/\pi^{n+1}B) = G(\varprojlim_n B/\pi^{n+1}B)$  vanish. Since the homomorphism  $B \rightarrow (B \otimes_R K) \times \varprojlim_n B/\pi^{n+1}B$  is faithful flat, the map  $G(B) \rightarrow G(B \otimes_R K) \times \varprojlim_n G(B/\pi^{n+1}B)$  is injective. It follows that the image of  $a$  in  $G(R)$  is the unit element, whence  $a \in F := \text{Ker}(G' \rightarrow G)$ . Consider the following commutative diagram

$$\begin{array}{ccccc} & & (A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}(B) & & \\ & \swarrow & \downarrow \alpha_1 & \searrow m_- & \\ A_{mY, \Sigma_{\square_m^d}}(B) \times A_{mY, \Sigma_{\square_m^d}}(B) & & & & A_{mY, \Sigma_{\square_m^d}}(B) \\ \downarrow \alpha_2 \times \alpha_2 & & \downarrow & & \downarrow \alpha_2 \\ Q_{mY, \Sigma_{\square_m^d}} \times Q_{mY, \Sigma_{\square_m^d}} & & (Q_{mY, \Sigma_{\square_m^d}} \times Q_{mY, \Sigma_{\square_m^d}})_{\square} & & Q_{mY, \Sigma_{\square_m^d}} \end{array}$$

where  $Q_{mY, \Sigma_{\square_m^d}}$  denotes the set

$$A_{mY, \Sigma_{\square_m^d}}(B \otimes K) \times \varprojlim_n A_{mY, \Sigma_{\square_m^d}}(B/\pi^{n+1})$$

and  $(Q_{mY, \Sigma_{\square_m^d}} \times Q_{mY, \Sigma_{\square_m^d}})_{\square}$  denotes the set

$$(A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}(B \otimes K) \times \varprojlim_n (A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}(B/\pi^{n+1})$$

Then we get  $\alpha_2(a) = \alpha_2(\langle \cdot, y \rangle \tilde{f})^{-1} \cdot \alpha_2(\tilde{g})$ , it follows that

$$\alpha_2(\tilde{g}) = \alpha_2(a) \cdot \alpha_2(\langle \cdot, y \rangle \tilde{f})$$

Here the group operation takes place in the bigger set

$$A_{mY, \Sigma_{\square_m^d}}(B \otimes K) \times \varprojlim_n T_{\log}^{(Y)}/(T \cdot mY)(B/\pi^{n+1})$$



which is obviously a group. But we know  $\alpha_2$  is injective, and  $F$  is an open subgroup of the constant finite  $S$ -group scheme  $(Y/H)_S$ , hence  $\langle \cdot, y \rangle \tilde{f}$  is equivalent to  $\tilde{g}$  under the action of  $F$ . It follows that  $f = g$ .  $\square$

LEMMA 4.4. *In the situation of 4.3,  $A(R)$  is a subgroup of  $A(R) \rightarrow A(R \otimes_R K) \times \varprojlim A(B/\pi^{n+1}B)$ .*

PROOF. Let  $f, g \in A(B)$ . By 4.2, we can assume that  $f, g$  come from  $\tilde{f}, \tilde{g} \in A_{mY, \Sigma_{\square_m^d}}$  for some  $m \geq 1$ , such that  $(\tilde{f}, \tilde{g})$  lies in  $(A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}$ . Write the group  $A(R \otimes_R K) \times \varprojlim A(B/\pi^{n+1}B)$  as  $W$ , we have the following commutative diagram

$$\begin{array}{ccccc}
 (\tilde{f}, \tilde{g}) \in & (A_{mY, \Sigma_{\square_m^d}} \times A_{mY, \Sigma_{\square_m^d}})_{\square}(B) & & & \\
 \parallel & \downarrow & \searrow m_- & & \\
 (\tilde{f}, \tilde{g}) \in & A_{mY, \Sigma_{\square_m^d}}(B) \times A_{mY, \Sigma_{\square_m^d}}(B) & & A_{mY, \Sigma_{\square_m^d}}(B) & \ni m_-(\tilde{f}, \tilde{g}) \\
 \downarrow & \downarrow & & \downarrow & \\
 (f, g) \in & A(B) \times A(B) & & A(B) & \\
 & \downarrow & & \downarrow & \\
 & W \times W & \xrightarrow{(a, b) \mapsto a^{-1} \cdot b} & W & 
 \end{array}$$

Hence as a subset of  $W$ ,  $A(B)$  is closed under the group operation of  $W$ , hence a subgroup of  $W$ .  $\square$

PROOF OF THEOREM 4.1, CONTINUED: Now we come to the proof of (b). Combining 4.3 and 4.4, by a limit argument which reduces the problem to noetherian rings, we get a unique group structure on  $A$  with required properties.

For (d), the injectivity of  $\alpha$  and the surjectivity of  $\beta$  are clear, so we are left to show the exactness in the middle. Further investigation of the definition of  $\alpha$  and  $\beta$ , we have  $\beta \cdot \alpha = 0$ . Now let  $f \in A(\text{Spec} B)$  for a noetherian fs log scheme  $\text{Spec} B$  over  $S$  with  $\beta(f) = 0$ . By (c), we know  $f \in A_Y(\text{Spec} B)$  with  $f_0 = f \times_S S_0 = 0$ , hence the image of  $f$  lies in the open subspace  $G$  of  $A_Y$ . This finishes the proof of (d).  $\square$

PROOF OF THEOREM 4.2: By 4.1 (a), the condition [KKN08a, 4.1.1] is satisfied. By 4.1 (d), the condition [KKN08a, 4.1.2] is satisfied. We are left to show the separateness condition [KKN08a, 4.1.3]. Let  $A'_K$  be the abelian variety with period lattice  $mY$  for a positive integer  $m$ , let  $A'$  be the log abelian variety corresponding to  $A'_K$ . By construction 4.3, we have a short exact sequence

$$0 \rightarrow Y/mY \rightarrow A' \rightarrow A \rightarrow 0$$

of abelian sheaves over  $(\text{fs}/S)$ . Now for  $U \in (\text{fs}/S)$ , given two morphisms  $f, g : U \rightarrow A$ , we have  $f, g$  come from sections  $f', g'$  of  $A_{mY, \Sigma_{\square_m^d}}$  étale locally for some  $m$ . By

the above short exact sequence, the equalizer  $E(f, g)$  is locally the disjoint union of  $E(f', a \cdot g')$  with  $a$  varying in  $Y/mY$ . The algebraic space  $A_{mY, \Sigma_{\square_m^d}}$  is separated over  $S$ , hence  $E(f', a \cdot g')$  is finite over  $U$ , so is  $E(f, g)$ . This finishes the proof.  $\square$

**4.3.** In the last subsection, we have associated a canonical log abelian variety  $A$  over  $S$  to any split totally degenerate abelian variety  $A_K$  over  $K$ . Let  $\text{STDAV}_K$  (resp.  $\text{LAV}_S$ ) denote the category of split totally degenerate abelian varieties over  $K$  (resp. log abelian varieties over  $S$ ). We regard this association as a map

$$\text{Deg} : \text{STDAV}_K \rightarrow \text{LAV}_S$$

In this subsection, we would like to show that  $\text{Deg}$  is actually a functor. In other words, we will associate to every homomorphism  $f : A_K \rightarrow A'_K$  in  $\text{STDAV}_K$  a homomorphism from  $A := \text{Deg}(A_K)$  to  $A' := \text{Deg}(A'_K)$ .

Let  $T$  (resp.  $T'$ ) and  $Y$  (resp.  $Y'$ ) be the Raynaud extension and period lattice of  $A_K$  (resp.  $A'_K$ ) respectively, and let  $Y \xrightarrow{u_K} T_K$  (resp.  $Y' \xrightarrow{u'_K} T'_K$ ) be the rigid analytic uniformisation of  $A_K$  (resp.  $A'_K$ ). Then we get a functorial homomorphism

$$\begin{array}{ccc} Y & \xrightarrow{f_{-1}} & Y' \\ \downarrow u_K & & \downarrow u'_K \\ T_K & \xrightarrow{f_0} & T'_K \end{array}$$

of 1-motives over  $K$ . This further gives the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f_{-1}} & Y' \\ \downarrow u_K & & \downarrow u'_K \\ E = \text{Hom}(X, \mathbb{Q}) & \longrightarrow & E' = \text{Hom}(X', \mathbb{Q}) \end{array}$$

after taking valuation maps.

Now in order to have a map from  $A$  to  $A'$ , we need to find  $A_{H, \Sigma}$  somewhere to go for each pair  $(H, \Sigma) \in \text{PolDecom}_Y$ . This comes down to find a pair  $(H', \Sigma' \in \text{PolDecom}_{Y'})$  such that  $f_{-1}(H) \subset H'$  and for any  $\sigma \in \Sigma$  there exists  $\sigma' \in \Sigma'$  with  $f_{-1}(\sigma) \subset \sigma'$ .

Let  $\overline{f_{-1}(Y)} := f_{-1}(E) \cap Y'$ , we have that  $Y' = \overline{f_{-1}(Y)} \oplus \tilde{Y}'$  for some subgroup  $\tilde{Y}' \leq Y'$  (we fix  $\tilde{Y}'$  from now on), and  $E' = f_{-1}(E) \oplus \tilde{E}'$  with  $\tilde{E}' := \tilde{Y}' \otimes \mathbb{Q}$ . After choosing a  $\mathbb{Z}$ -basis of  $\tilde{Y}'$ , we make a  $\tilde{Y}'$ -admissible polytope decomposition  $\Lambda$  of  $\tilde{E}'$  by the cubes with respect to this basis. Note that  $f_{-1}(\Sigma)$  is an  $f_{-1}(H)$ -admissible polytope decomposition of  $f_{-1}(E)$ . We make a polytope decomposition  $\Sigma'$  by the product of  $f_{-1}(\Sigma)$  and  $\Lambda$ . Let  $H' = f_{-1}(H) + \tilde{Y}'$ , it is clear that  $\Sigma'$  is  $H'$ -admissible. So we get a pair

$$(4.5) \quad (H', \Sigma') \in \text{PolDecom}_{Y'}$$

with required properties. It follows that we get a morphism  $A_{H, \Sigma} \rightarrow A'_{H', \Sigma'}$  of proper algebraic  $S$ -spaces. We map  $A_{H, \Sigma}$  into  $A'$  by the composition  $A_{H, \Sigma} \rightarrow A'_{H', \Sigma'} \rightarrow A'$ .

This is well-defined. If we are given another such pair  $(H'_1, \Sigma'_1)$ , then the pair  $(H' \cap H'_1, \Sigma' \cap \Sigma'_1)$  is a third such pair. Hence we get the following commutative diagram

$$\begin{array}{ccccc} & & A_{H, \Sigma} & & \\ & \swarrow & \downarrow & \searrow & \\ A'_{H', \Sigma'} & \longleftarrow & A'_{H' \cap H'_1, \Sigma' \cap \Sigma'_1} & \longrightarrow & A'_{H'_1, \Sigma'_1} \end{array}$$

which guarantees the map  $A_{H, \Sigma} \rightarrow A'$  being well-defined. We denote it by  $\text{Deg}(f)_{H, \Sigma}$ .

Now we want show that the collection  $(\text{Deg}(f)_{H, \Sigma})_{(H, \Sigma)}$  glues to a map from  $A$  to  $A'$ . In other words, the collection  $(\text{Deg}(f)_{H, \Sigma})_{(H, \Sigma)}$  is compatible with the equivalence relations in (4.3). Given a map  $(H_1, \Sigma_1) \rightarrow (H_2, \Sigma_2)$  in  $\text{PolDecom}_Y$ , hence  $H_1$  is a subgroup of  $H_2$  and  $\Sigma_1$  is a subdivision of  $\Sigma_2$ . The construction in 4.5 is compatible with  $(H_1, \Sigma_1) \rightarrow (H_2, \Sigma_2)$ , hence we get a commutative diagram

$$\begin{array}{ccc} A_{H_1, \Sigma_1} & \longrightarrow & A_{H_2, \Sigma_2} \\ \downarrow & & \downarrow \\ A'_{H'_1, \Sigma'_1} & \longrightarrow & A'_{H'_2, \Sigma'_2} \end{array}$$

This show the compatibility for the first kind of equivalences in 4.3. Similar argument works for the second kind of equivalences in 4.3 too. Hence the collection  $(\text{Deg}(f)_{H, \Sigma})_{(H, \Sigma)}$  glues into a map  $\text{Deg}(f) : A \rightarrow A'$ . The construction of  $\text{Deg}$  is clearly functorial, whence the following theorem.

**THEOREM 4.3.** *The association of a log abelian variety to any split totally degenerate abelian variety over  $K$  gives rise to a functor  $\text{Deg} : \text{STDAV}_K \rightarrow \text{LAV}_S$ . Moreover, the functor  $A \rightarrow A_K := A \times_S \text{Spec} K$  is left inverse to  $\text{Deg}$  and  $\text{Deg}$  is fully faithful.*

**PROOF.** The only thing needed to check is the fullness. Given any two log abelian varieties  $A_1$  and  $A_2$  over  $S$ , let

$$0 \rightarrow G_1 \rightarrow A_1 \rightarrow \mathcal{H}om_S(X_1, \mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y_1)}/\bar{Y}_1 \rightarrow 0$$

and

$$0 \rightarrow G_2 \rightarrow A_2 \rightarrow \mathcal{H}om_S(X_2, \mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y_2)}/\bar{Y}_2 \rightarrow 0$$

be the corresponding short exact sequences associated to log abelian varieties (see [KKN08a, Def. 4.1, 4.1.2]). Let  $f$  be a homomorphism  $f : A_1 \rightarrow A_2$ , then  $f$  induces a homomorphism from  $\tilde{f} : G_1 \rightarrow \mathcal{H}om_S(X_2, \mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y_2)}/\bar{Y}_2$ . Since  $\tilde{f}$  is zero by [KKN08a, 9.2],  $f$  induces homomorphisms  $g : G_1 \rightarrow G_2$  and  $h : \mathcal{H}om_S(X_1, \mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y_1)}/\bar{Y}_1 \rightarrow \mathcal{H}om_S(X_2, \mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y_2)}/\bar{Y}_2$ . It is clear that the pair  $(g, h)$  determines  $f$ . It is also clear that  $g_K = f_K$  determines  $g$ . We are going to show that  $g$  determines  $h$  in the case that both  $A_1$  and  $A_2$  lie in the image of  $\text{Deg}$ . Then  $f_K$  determines  $f$  and the fullness follows.

Now suppose that the  $A_i$ 's lie in the image of the functor  $\text{Deg}$ , and they are constructed out of some bilinear pairings  $\langle, \rangle_i : X_i \times Y_i \rightarrow \mathbb{G}_{m, \log}$ , see (4.1). The

pairing  $\langle, \rangle_i$  gives rise to  $\mathcal{H}om_S(X_i, \mathbb{G}_{m, \log}/\mathbb{G}_m)^{(Y_i)}/\bar{Y}_i$ , and it corresponds to a log 1-motive  $[Y_i \rightarrow (T_i)_{\log}]$ . The log 1-motive  $[Y_i \rightarrow (T_i)_{\log}]$  can be recovered from  $G_i$  through the equivalence of categories in [FC90, Chap. III, Prop. 6.4]. The map  $g$  is nothing but a map between two elements of  $\text{DEG}_{\text{ample}}$  forgetting the invertible sheaves, while the map  $h$  can be obtained from the map of log 1-motives corresponding to  $g$ . Hence the map  $h$  is determined by the map  $g$ . This finishes the proof.  $\square$

REMARK 4.3. In future we hope to generalise the results in 4.2 and 4.3 to all abelian varieties  $A_K$  over  $K$ . In the case that the Raynaud extension of  $A_K$  has split torus part, the result follows probably from the split totally degenerate case with the help of contracted product. We also expect that the functor  $\text{Deg}$  extends to an equivalence between the category of abelian varieties over  $K$  and the category of log abelian varieties over  $S$ . Such an equivalence would justify the motto “log abelian varieties are canonical degenerations of abelian varieties”.

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